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MOYAL NONCOMMUTATIVE INTEGRABILITY AND THE BURGERS-KDV MAPPING

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Abstract

The Moyal momentum algebra, studied in [20, 21], is once again used to discuss some important aspects of NC integrable models and $2d$ conformal field theories. Among the results presented, we setup algebraic structures and makes useful convention notations leading to extract non trivial properties of the Moyal momentum algebra. We study also the Lax pair building mechanism for particular examples namely, the noncommutative KdV and Burgers systems. We show in a crucial step that these two systems are mapped to each others through the following crucial mapping $\partial_{t_2} \hookrightarrow \partial_{t_3} \equiv \partial_{t_2}\partial_x + \alpha\partial_x^3$. This makes a strong constraint on the NC Burgers system which corresponds to linearizing its associated differential equation. From the CFT's point of view, this constraint equation is nothing but the analogue of the conservation law of the conformal current. We believe that the considered mapping might help to bring new insights towards understanding the integrability of noncommutative $2d$-systems.

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1 Introduction

Recently there has been a revival interest in the noncommutativity of coordinates in string theory and D-brane physics[1, 2, 3, 4, 5, 6]. This interest is known to concern also noncommutative quantum mechanics and noncommutative field theories [7, 8]. The sharing property between all the above interesting areas of research is that the corresponding space exhibits the following structure

$$[x_i, x_j]_{*'} = i\theta_{ij} \tag{1}$$

where x_i are non-commuting coordinates which can describe also the space-time coordinates operators and θ_{ij} is a constant antisymmetric tensor. Quantum field theories living on this space are necessarily noncommutative field theories. Their formulation is simply obtained when the algebra (1) is realized in the space of fields (functions) by means of the Moyal bracket according to which the usual product of functions is replaced by the star-product as follows [9],

$$(f * g)(x) = f(x)e^{\frac{i}{2}\theta^{ab}\overleftarrow{\partial_a}\overrightarrow{\partial_b}}g(x), \tag{2}$$

The link with string theory consist on the correspondence between the θ^{ij}-constant parameter and the constant antisymmetric two-form potential B^{ij} on the brane as follows [1],

$$\theta^{ij} = (\frac{1}{B})^{ij}, \tag{3}$$

such that in the presence of this B-field, the end points of an open string become noncommutative on the D-brane. The same interest on the noncommutative geometry is exhibited by the $(1 + 1)$-dimensional integrable models [10] which are intimately connected to conformal field theories [11] and their underlying lower ($s \leq 2$) [12, 13, 14, 15, 16] and higher ($s \geq 2$) [17, 18] spin symmetries.

Non trivial integrable models are, in general situations, based on nonlinear differential equations which may be solvable. However, few nonlinear differential equations which are integrable. This property is traced to the fact that solving a nonlinear system is not an easy job in most of the considered cases.

As the notion of integrability, of a given system, is one of important physical requirements, one have to overcome the difficulties of nonlinearity by adopting adequate technics. In this sense, one can anticipate a first definition of integrability as been the possibility to linearize the associated nonlinear differential equation. The famous approach of linearizing a physical system is given by the well known Lax technics related to the inverse scattering transformation[19]. The idea consist in assuming the existence of a pair of operators, L and B satisfying the following linear equation

$$\partial_t\mathcal{L} = [\mathcal{L}, B]. \tag{4}$$

Usually the application of the inverse scattering transformation method to an evolution equation is based on the Lax representation. The linear Lax equation (4) describes then an evolution equation of the differential operators L and B with $[\mathcal{L};\mathcal{B}]$ is their commutator.

One of the principal objectives of this work is study the Lax representation in the case of noncommutative systems. This is an important mathematical and physical issue expected to shed more lights on the notion of integrability. We will make some consistent assumptions shown to be essential in in deriving the Lax pair of special noncommutative integrable systems.

One need to extract new properties of the integrability in the case of noncommutative spaces. The Lax representation provides a sophisticated means to achieve such a goal. It's commonly known that the existence of the Lax pair gives a significant sign of integrability. The importance of noncommutative extensions find then a good motivation through this issue. Accordingly, we will consider Lax formalism as the way to test the integrability of the noncommutative systems. In the same philosophy, the intervention of conformal symmetry is planned to reinforce our analysis and to push our research of the source of the integrability ahead.

The principal prototype examples considered in this study are the KdV and Burgers systems. The originality of the present work deals with the possibility to connect these noncommutative systems through a consistent mapping that we will setup. Before describing the essential of this mapping transformation, let's first present briefly the content the successive sections.

We give in *section2* some generalities on the convention notations that we use and on the algebraic structures of the Moyal momentum algebra $\widehat{\Sigma}$ introduced in [20, 21].

In *section 3* we present an explicit description of the Moyal momentum algebra $\widehat{\Sigma}$ and show how it can relies to $2d$-CFT. *section4* is devoted to some important implications of the Moyal momentum algebra on generalized KdV hierarchies. We will concentrate on the noncommutative sl_2 KdV equation and the noncommutative version of the Burgers equation and their Lax representations. The derived properties may naturally be derived for the more general case namely the sl_n KdV hierarchy.

The *Section 5* is devoted to a set up of the Lax pair representation of special noncommutative integrable systems namely the noncommutative KdV and Burgers systems. Here we present a systematic study of noncommutative generating Lax pair operators in the Moyal momentum framework. The essential results deals with the derivation of the noncommutative KdV and Burgers systems. Concerning the noncommutative derived KdV system, this is an integrable model due to its underlying conformal symmetry.

Going in the same lines of our previous works on noncommutative geometry *à* la Moyal, we try in *section 6* to study a possible relation between the NC KdV and the NC Burgers systems. Several important facts to support this possibility are discussed.

2 Noncommutativity *à* la Moyal: Generalities

2.1 Basic Definitions

1. We start first by recalling that the functions often involved in the $2d$-phase-space are arbitrary functions which we generally indicate by $f(x,p)$ where the variable x stands for the space coordinate while p describes the momentum coordinate.
2. With respect to this phase space, we have to precise that the constants f_0 are defined such that

$$\partial_x f_0 = 0 = \partial_p f_0. \tag{5}$$

3. The functions $u_i(x,t)$ depending on an infinite set of variables $t_1 = x, t_2, t_3, ...$, do not depend on momentum coordinates, which means

$$\partial_p u_i(x,t) = 0, \tag{6}$$

where the index i, describes the conformal weight of the field $u_i(x,t)$. These functions can be considered in the complex language framework as being the analytic (conformal) fields of conformal spin $i = 1, 2, ...$
4. Other objects usually used are the ones given by

$$u_i(x,t) \star p^j, \tag{7}$$

which are objects of conformal weight $(i+j)$ living on the non-commutative space parametrized by θ. Through this work, we will use the following convention notations $[u_i] = i$, $[\theta] = 0$ and $[p] = [\partial_x] = -[x] = 1$, where the symbol $[\quad]$ stands for the conformal dimension of the used objects.
5. The star product law, defining the multiplication of objects in the non-commutative phase space, is given by the following expression

$$f(x,p) \star g(x,p) = \sum_{s=0}^{\infty}\sum_{i=0}^{s} \frac{\theta^s}{s!}(-)^i c_s^i (\partial_x^i \partial_p^{s-i} f)(\partial_x^{s-i}\partial_p^i g), \tag{8}$$

with $c_s^i = \frac{s!}{i!(s-i)!}$.
6. The conventional Moyal bracket is defined as

$$\{f(x,p), g(x,p)\}_\theta = \frac{f \star g - g \star f}{2\theta}, \tag{9}$$

where θ is the noncommutative parameter, considered as a constant in this approach[2].

7. To distinguish the classical objects from the θ-deformed ones, we consider the following convention notations [20, 21]:

a)$\widehat{\Sigma}_m^{(r,s)}$: This is the space of momentum differential operators of conformal weight m and degrees (r, s) with $r \leq s$. Typical operators of this space are given by

$$\sum_{i=r}^{s} u_{m-i} \star p^i. \tag{10}$$

b)$\widehat{\Sigma}_m^{(0,0)}$: This is the space of functions of conformal weight m; $m \in Z$, which may depend on the parameter θ. It coincides in the classical limit, $\theta \to \theta_l$[3]

, with the ring of analytic fields involved into the construction of conformal symmetry and w-extensions.

c)$\widehat{\Sigma}_m^{(k,k)}$: Is the space of momentum operators type,

$$u_{m-k} \star p^k. \tag{11}$$

d)θ-**Residue operation:** $\widehat{Res}$

$$\widehat{Res}(f \star p^{-1}) = f. \tag{12}$$

2.2 The Moyal Momentum algebra $\widehat{\Sigma}(\theta)$:

This is the algebra based on arbitrary momentum differential operators of arbitrary conformal weight m and arbitrary degrees (r, s). Its obtained by summing over all the allowed values of spin (conformal weight) and degrees in the following way:

$$\widehat{\Sigma}(\theta) = \oplus_{r \leq s} \oplus_{m \in Z} \widehat{\Sigma}_m^{(r,s)}. \tag{13}$$

$\widehat{\Sigma}(\theta)$ is an infinite dimensional momentum algebra which is closed under the Moyal bracket without any condition. A remarkable property of this space is the possibility to introduce six infinite dimensional classes of momentum sub-algebras related to each other by special duality relations. These classes of algebras are given by $\widehat{\Sigma}_s^{\pm}$, with $s = 0, +, -$ describing respectively the different values of the conformal spin which can be zero, positive or negative. The $\pm$ upper indices stand for the values of the degrees quantum numbers, for more details see [16, 20, 21].

[2]For an application of non constant θ parameter, see for instance [22] and references therein

[3]Usually the standard limit is taken such that $\theta_{limit} = 0$. In the present analysis, the standard limit is shifted by $\frac{1}{2}$ such that $\theta_l \to \theta_{limit} + \frac{1}{2}$. Thus taking the standard limit to be $\theta_{limit} = 0$ is equivalent to set $\theta_l = \frac{1}{2}$. The origin of this shift belongs to the consistent non commutative w_θ^3-Zamolodchikov algebra construction [20, 21]

2.2.1 Algebraic structure of The space $\widehat{\Sigma}_m^{(r,s)}$

To start let's precise that this space contains momentum operators of fixed conformal spin m and degrees (r,s), type

$$\mathcal{L}_m^{(r,s)}(u) = \sum_{i=r}^{s} u_{m-i}(x) \star p^i, \tag{14}$$

These are θ-differentials whose operator character is inherited from the star product law defined as in (8).

Using this relation, it is now important to precise how the momentum operators act on arbitrary functions $f(x,p)$ via the star product.

Performing computations based on relation (8), we find the following θ- Leibnitz rules:

$$p^n \star f(x,p) = \sum_{s=0}^{n} \theta^s c_n^s f^{(s)}(x,p) p^{n-s}, \tag{15}$$

and

$$p^{-n} \star f(x,p) = \sum_{s=0}^{\infty} (-)^s \theta^s c_{n+s-1}^s f^{(s)}(x,p) p^{-n-s}, \tag{16}$$

where $f^{(s)} = \partial_x^s f$ is the prime derivative. We also find the following expressions for the Moyal bracket:

$$\begin{aligned} \{p^n, f\}_\theta &= \textstyle\sum_{s=0}^{n} \theta^{s-1} c_n^s \{\frac{1-(-)^s}{2}\} f^s p^{n-s}, \\ \{p^{-n}, f\}_\theta &= \textstyle\sum_{s=0}^{\infty} \theta^{s-1} c_{s+n-1}^s \{\frac{(-)^s-1}{2}\} f^s p^{-n-s}, \end{aligned} \tag{17}$$

Special Moyal brackets are given by

$$\begin{aligned} \{p, x\}_\theta &= 1 \\ \{p^{-1}, x\}_\theta &= -p^{-2} \end{aligned} \tag{18}$$

With the derived the Leibnitz rules for the momentum operators, we can also remark that the momentum operators p^i satisfy the algebra

$$p^n \star p^m = p^{n+m}. \tag{19}$$

which ensures the suspected rule

$$\begin{aligned} p^n \star (p^{-n} \star f) &= f \\ (f \star p^{-n}) \star p^n &= f. \end{aligned} \tag{20}$$

2.2.2 Further Algebraic Properties of $\widehat{\Sigma}_m^{(r,s)}$:

An important algebraic property of the space $\widehat{\Sigma}_m^{(r,s)}$ is that it may decomposes into the underlying subspaces as

$$\widehat{\Sigma}_m^{(r,s)} = \oplus_{k=r}^{s} \widehat{\Sigma}_m^{(k,k)}(\theta) \tag{21}$$

where $\widehat{\Sigma}_m^{(k,k)}$ are unidimensional subspaces containing prototype elements of kind $u_{m-k} \star p^k$ or $p^k \star u_{m-k}$. Using the θ-Leibniz rule, we can write, for fixed value of k:

$$\widehat{\Sigma}_m^{(k,k)} \equiv \Sigma_m^{(k,k)} \oplus \theta \Sigma_m^{(k-1,k-1)} \oplus \theta^2 \Sigma_m^{(k-2,k-2)} \oplus ... \tag{22}$$

where $\Sigma_m^{(k,k)}$ is the standard one dimensional sub-space of Laurent series objects $u_{m-k}p^k$ considered also as the $(\theta = 0)$-limit of $\widehat{\Sigma}_m^{(k,k)}$.

This property can be summarized as follows

$$\begin{aligned} \widehat{\Sigma}_m^{(r,s)} &= \oplus_{k=r}^{s} \widehat{\Sigma}_m^{(k,k)}(\theta) \\ &= \oplus_{k=r}^{s} \oplus_{l=0}^{k} \theta^l \Sigma_m^{(k-l,k-l)} \end{aligned} \tag{23}$$

Furthermore, the unidimensional subspaces $\widehat{\Sigma}_m^{(k,k)}$ can be written formally as

$$\widehat{\Sigma}_m^{(k,k)} \equiv p^k \star \Sigma_{m-k}^{(0,0)}. \tag{24}$$

where $\widehat{\Sigma}_m^{(0,0)} \equiv \Sigma_m^{(0,0)}$ is nothing but the ring of analytic fields u_m of conformal spin $m \in Z$ satisfying

$$u_i \star u_j = u_i.u_j \tag{25}$$

Another property concerning the space $\widehat{\Sigma}_m^{(r,s)}$ is its non closure under the action of the Moyal bracket since we have;

$$\{.,.\}_\theta : \widehat{\Sigma}_m^{(r,s)} \star \widehat{\Sigma}_m^{(r,s)} \longrightarrow \widehat{\Sigma}_{2m}^{(r,2s-1)} \tag{26}$$

Imposing the closure, one gets strong constraints on the integers m, r and s namely

$$\begin{array}{rcl} m & = & 0 \\ r \leq & s & \leq 1 \end{array} \tag{27}$$

With these constraint equations, the sub-spaces $\widehat{\Sigma}_m^{(r,s)}$ exhibit then a Lie algebra structure since the $\star$-product is associative.

2.2.3 Residue Duality in $\widehat{\Sigma}_m^{(r,s)}$:

The sub-space $\widehat{\Sigma}_m^{(r,s)}$ is characterized by the existence of a residue operation that we denote as $\widehat{Res}$ and which acts as follows

$$\begin{aligned} \widehat{Res}(u_k \star p^{-k}) &= (u_k \star p^{-k})\delta_{k-1,0} \\ &= u_1 \delta_{k-1,0} \end{aligned} \tag{28}$$

This result coincides with the standard residue operation: Res, acting on the sub-space $\Sigma_m^{(r,s)}$:

$$Res(u_1.p^{-1}) = u_1 \tag{29}$$

We thus have two type of residues $\widehat{Res}$ and Res acting on two different spaces $\widehat{\Sigma}_m^{(r,s)}$ and $\Sigma_m^{(r,s)}$ but with value on the same ring $\Sigma_{m+1}^{(0,0)}$. This Property is summarized as follows:

$$\begin{tikzcd} \widehat{\Sigma}_m^{(r,s)} \arrow[rr, "\theta=0"] \arrow[dr, "\widehat{Res}"'] & & \Sigma_m^{(r,s)} \arrow[dl, "Res"] \\ & \Sigma_{m+1}^{(0,0)} & \end{tikzcd} \tag{30}$$

We learn from this diagram that the residue operation exhibits a conformal spin quantum number equal to 1 as it maps objects of conformal spin m to the space $\widehat{\Sigma}_{m+1}^{(0,0)}$. The other important property of the residue operation is that it acts only on the $\widehat{\Sigma}_m^{(-1,-1)}$

With respect to the previous residue operation, we define on $\widehat{\Sigma}$ the following degrees pairing product

$$(.,.) : \widehat{\Sigma}_m^{(r,s)} \star \widehat{\Sigma}_n^{(-s-1,-r-1)} \longrightarrow \Sigma_{m+n+1}^{(0,0)} \tag{31}$$

such that

$$\left(\mathcal{L}_m^{(r,s)}(u), \tilde{\mathcal{L}}_n^{(\alpha,\beta)}(v)\right) = \delta_{\alpha+s+1,0}\delta_{\beta+r+1,0}\widehat{Res}\left[\mathcal{L}_m^{(r,s)}(u) \star \tilde{\mathcal{L}}_n^{(\alpha,\beta)}(v)\right], \tag{32}$$

showing that the spaces $\widehat{\Sigma}_m^{(r,s)}$ and $\widehat{\Sigma}_n^{(-s-1,-r-1)}$ are $\widehat{Res}$-dual as $\Sigma_m^{(r,s)}$ and $\Sigma_n^{(-s-1,-r-1)}$ are dual with respect to the standard Res-operation.

2.2.4 The Lie algebra $\widehat{\Sigma}_0^{(-\infty,1)}$ section of $\widehat{\Sigma}_m^{(r,s)}$

As discussed previously, the subspaces $\widehat{\Sigma}_m^{(r,s)}$ exhibit a Lie algebra structure with respect to the Moyal bracket once the spin-degrees constraints (27) are considered. With these conditions one should note that the huge Lie algebra that we can extract from the space $\widehat{\Sigma}_m^{(r,s)}$ consists on the space $\widehat{\Sigma}_0^{(-\infty,1)}$ having the remarkable space decomposition

$$\widehat{\Sigma}_0^{(-\infty,1)} = \widehat{\Sigma}_0^{(-\infty,-1)} \oplus \widehat{\Sigma}_0^{(0,1)}, \tag{33}$$

where $\widehat{\Sigma}_0^{(-\infty,-1)}$ describes the Lie algebra of pure non local momentum operators and $\widehat{\Sigma}_0^{(0,1)}$ is the Lie algebra of local Lorentz scalar momentum operators $\mathcal{L}\ _0(u) = u_{-1} \star p + u_0$. The latter can splits as follows

$$\widehat{\Sigma}_0^{(0,1)} = \widehat{\Sigma}_0^{(0,0)} \oplus \widehat{\Sigma}_0^{(1,1)}, \tag{34}$$

where $\widehat{\Sigma}_0^{(1,1)}$ is the Lie algebra of vector momentum fields $J_0(u) = u_{-1} \star p$ which are also elements of $\Sigma_0^{(0,1)}$.

As a prototype example, consider

$$\begin{aligned} \mathcal{L}_u &= u_{-1} \star p + u_0 \\ \mathcal{L}_v &= v_{-1} \star p + v_0 \end{aligned} \tag{35}$$

be two elements of $\widehat{\Sigma}_0^{(0,1)}$. Straightforward computations lead to:

$$\{\mathcal{L}_u, \mathcal{L}_v\}_\theta = \mathcal{L}_w \tag{36}$$

with

$$
\begin{aligned}
\mathcal{L}_w &= w_{-1} \star p + w_0 \\
&= \{u_{-1}v'_{-1} - u'_{-1}v_{-1}\} \star p + \{u_{-1}v'_0 - u'_0 v_{-1}\}.
\end{aligned} \tag{37}
$$

Forgetting about the fields (*of vanishing conformal spin*) belonging to $\Sigma_0^{(0,0)}$ is equivalent to consider the coset space

$$
\widehat{\Sigma}_0^{(1,1)} \equiv \widehat{\Sigma}_0^{(0,1)} / \Sigma_0^{(0,0)} \tag{38}
$$

one obtain the $Diff(S^1)$ momentum algebra of vector fields $J_0(u) = u_{-1} \star p$ namely

$$
\{J_0(u), J_0(v)\}_\theta = J_0(w) \tag{39}
$$

with $w_{-1} = u_{-1}v'_{-1} - u'_{-1}v_{-1}$.

The extension of these results to non local momentum operators is natural. In fact, one easily show that the previous Lie algebras are simply sub-algebras of the huge momentum space $\widehat{\Sigma}_0^{(-\infty,1)}$. For a given $0 \leq k \leq 1$, we have

$$
\widehat{\Sigma}_0^{(0,1)} \subset \widehat{\Sigma}_0^{(-\infty,k)} \subset \widehat{\Sigma}_0^{(-\infty,1)} \tag{40}
$$

and by virtue of (47)

$$
\{\widehat{\Sigma}_0^{(-\infty,k)}, \widehat{\Sigma}_0^{(0,1)}\}_\theta \subset \widehat{\Sigma}_0^{(-\infty,k)} \subset \widehat{\Sigma}_0^{(-\infty,1)} \tag{41}
$$

and for $-\infty < p \leq q \leq 1$

$$
\{\widehat{\Sigma}_0^{(-\infty,p)}, \widehat{\Sigma}_0^{(-\infty,q)}\}_\theta \subset \widehat{\Sigma}_0^{(-\infty,p+q-1)} \tag{42}
$$

These Moyal bracket expressions show in turn that all the subspaces $\widehat{\Sigma}_0^{(p,q)}$ with $-\infty < p \leq q \leq 1$ are ideals of of $\widehat{\Sigma}_0^{(-\infty,1)}$.

3 The $sl_n - \widehat{\Sigma}_n^{(0,n)}(\theta)$ NC Algebra and $2d$ CFT

The algebra $sl_n - \widehat{\Sigma}_n^{(0,n)}(\theta)$ describes simply the coset space $\widehat{\Sigma}_n^{(0,n)}/\widehat{\Sigma}_n^{(1,1)}$ of sl_n-Lax operators given by

$$
\mathcal{L}_n(u) = p^n + \sum_{i=0}^{n-2} u_{n-i} \star p^i \tag{43}
$$

where we have set $u_0 = 1$ and $u_1 = 0$. This is a natural generalization of the well known sl_2-momentum Lax operator

$$
\mathcal{L}_2 = p^2 + u_2 \tag{44}
$$

associated to the θ-KdV integrable hierarchy that we will discuss later.

We have to underline that the sl_n-momentum Lax operators play a central role in the study of integrable models and more particularly in deriving higher conformal spin algebras (w_θ-algebras) from the θ-extended Gelfand-Dickey second Hamiltonian structure [20, 21, 23]. Since they are

also important in recovering $2d$ conformal field theories via the Miura transformation, we guess that its possible to extend this property, in a natural way, to the non-commutative case and consider the θ-deformed analogue of the well known $2d$ conformal models namely: the sl_2-Liouville field theory and its sl_n-Toda extensions and also the Wess-Zumino- Novikov-Witten conformal model.

As an example consider the θ-KdV momentum Lax operator that we can write as

$$\begin{aligned} \mathcal{L}_2 &= p^2 + u_2 \\ &= (p + \phi') \star (p - \phi') \end{aligned} \tag{45}$$

where ϕ is a Lorentz scalar field. As a result we have

$$u_2 = -\phi'^2 - 2\theta\phi'' \tag{46}$$

which is nothing but the θ analogue of classical stress energy momentum tensor of $2d$ conformal Liouville field theory. Using $2d$ complex coordinates language, we can write

$$T_\theta(z) \equiv u_2(z) = -2\theta\partial^2\phi - (\partial\phi)^2 \tag{47}$$

with $\partial \equiv \partial_z \equiv \frac{\partial}{\partial z}$. The conservation for this conformal current namely $\bar{\partial}T(z) = 0$, leads to write the following θ-Liouville equation of motion

$$\partial\bar{\partial}\phi = \frac{2}{\theta}e^{-\frac{1}{\theta}\phi} \tag{48}$$

associated to the two dimensional θ-Liouville action

$$S = \int d^2z \left(\frac{1}{2}\partial\phi \star \bar{\partial}\phi + e^{-\frac{1}{\theta}\phi}\right) \tag{49}$$

with $\partial\phi \star \bar{\partial}\phi = \partial\phi\bar{\partial}\phi$.

Note by the way that we may interpret the inverse $\frac{1}{\theta}$ of the non-commutative θ-parameter as being the analogue of the Cartan matrix of sl_2 because in the classical limit this Cartan matrix is known to be $(a_{ij}) = 2$ once the classical limit $\theta_l \to \frac{1}{2}$ is considered.
Another important point is that we know from the standard $2d$ CFT [11] that the object T_{zz} satisfying

$$T_{zz} = -\frac{1}{4}(\partial\phi)^2 + i\alpha_0\partial^2\phi \tag{50}$$

is nothing but the Feigin-Fuchs representation of the conserved current generating the conformal invariance of a quantum conformal model with the central charge $c = (1 - 24{\alpha_0}^2)$.

Using this standard result, we can conclude that the θ-conformal current that we derive in (47) (with the rescaling $\phi \equiv \frac{1}{2}\phi$) is associated to the θ-Liouville model having the central charge

$$c_\theta = (1 + 24\theta^2) \tag{51}$$

where the non-commutative parameter is shown to coincide with α_0 as follows $\theta = -i\alpha_0$.

The analysis that we use to derive the θ-Liouville equation and its central charge a la Feigin-Fuchs, can be generalized to higher conformal spin Toda field theories associated to sl_n symmetry with $(n-1)-$conserved currents $T(z), w_3, w_4, ...w_n$.

Finally we note that all the properties discussed above may be generalized to the sl_n case. This is an explicit proof of the importance of the algebraic structure inherited from the Moyal momentum algebra. Actually we showed how these algebra may leads to extend, in a successful, way all the important properties of $2d$ CFT theories. We will present in the next section some other applications of the momentum algebra in θ-integrable KdV hierarchies

4 Noncommutative sl_n KdV-hierarchy

The aim of this section is to present some results related to the θ-KdV hierarchy and that are explicitly derived in [20, 21]. Using our convention notations and the analysis that we developed previously, we will perform hard algebraic computations and derive the θ-KdV hierarchy.
The concerned computations are very hard and difficult to realize in the general case. For this reason, we concentrate simply on the first orders of the hierarchy namely the sl_2-KdV and sl_3-Boussinesq θ-integrable hierarchies.

The study concerns some results obtained in [20, 21] generalizing the ones obtained in [23] by increasing the order of computations a fact which leads to discover more important properties. As an original result, we were able to build the θ-deformed sl_3-Boussinesq hierarchy and derive the associated θ-flows.

4.1 sl_2-KdV hierarchy

Let's consider the sl_2-momentum Lax operator

$$\mathcal{L}_2 = p^2 + u_2 \tag{52}$$

whose 2th root is given by

$$\begin{aligned} \mathcal{L}^{\frac{1}{2}} &= \Sigma_{i=-1} b_{i+1} \star p^{-i} \\ &= \Sigma_{i=-1} a_{i+1} p^{-i} \end{aligned} \tag{53}$$

This 2th root of $\mathcal{L}_2$ is an object of conformal spin $[\mathcal{L}^{\frac{1}{2}}] = 1$ that plays a central role in the derivation of the θ-Lax evolutions equations. In the spirit to contribute much more to this sl_2-KdV hierarchy, it was important for us to make contact with previous works in literature [23]. Performing lengthy but straightforward calculations we compute the coefficients b_{i+1} of $\mathcal{L}^{\frac{1}{2}}$ up

to $i = 7$ given by [4]:

$$\begin{aligned}
b_0 &= 1 \\
b_1 &= 0 \\
b_2 &= \tfrac{1}{2}u \\
b_3 &= -\tfrac{1}{2}\theta u' \\
b_4 &= -\tfrac{1}{8}u^2 + \tfrac{1}{2}\theta^2 u'' \\
b_5 &= -\tfrac{1}{2}\theta^3 u''' + \tfrac{3}{4}\theta u u' \\
b_6 &= \tfrac{1}{16}u^3 - \tfrac{7}{4}\theta^2 u u'' - \tfrac{11}{8}\theta^2 (u')^2 + \tfrac{1}{2}\theta^4 u'''' \\
b_7 &= -\tfrac{15}{16}\theta u^2 u' + \theta^3(\tfrac{15}{2}u''u' + \tfrac{15}{4}uu''') - \tfrac{1}{2}\theta^5 u^{(5)} \\
b_8 &= -\tfrac{5}{128}u^4 + \theta^2(\tfrac{55}{16}u''u^2 + \tfrac{85}{16}uu'^2) - \theta^4(\tfrac{31}{4}uu'''' + \tfrac{91}{8}u''^2 + \tfrac{37}{2}u'u''') + \tfrac{1}{2}\theta^6 u^{(6)}
\end{aligned} \tag{54}$$

and

$$\begin{aligned}
b_9 &= \tfrac{35}{32}\theta u^3 u' - \tfrac{175}{4}\theta^3 \left(uu'u'' + \tfrac{1}{4}u'^3 + \tfrac{1}{4}u^2 u'''\right) + \tfrac{7}{4}\theta^5 \left(9uu^{(5)} + 25u^{(4)}u' + 35u'''u''\right) \\
&\quad -\tfrac{1}{2}\theta^7 u^{(7)} \\
b_{10} &= \tfrac{7}{256}u^5 - \tfrac{35}{32}\theta^2 \left(\tfrac{23}{2}u^2 u'^2 + 5u^3 u''\right) + \tfrac{7}{4}\theta^4 \left(\tfrac{73}{4}u^2 u^{(4)} + \tfrac{227}{4}uu''^2 + \tfrac{337}{4}u''u'^2 + 89uu'u'''\right) \\
&\quad -\tfrac{3}{4}\theta^6 \left(\tfrac{631}{3}u'' u^{(4)} + 233uu'''^2 + 135u'u^{(5)}\right) + \tfrac{1}{2}\theta^8 u^{(8)}
\end{aligned} \tag{55}$$

These results are obtained by using the identification $\mathcal{L}_2 = \mathcal{L}^{\frac{1}{2}} * \mathcal{L}^{\frac{1}{2}}$. Note that by virtue of (53), the coefficients a_{i+1} are shown to be functions of b_{i+1} and their derivatives in the following way

$$a_{i+1} = \sum_{s=0}^{i-1} \theta^s c_{i-1}^s b_{i+1-s}^{(s)}, \tag{56}$$

[4]The authors of [23] present explicit computations of the coefficients a_{i+1} and omit the b_{i+1} ones. Here, we give explicit computation of both of them

Substituting the derived expressions of b_{i+1} (54-55) into (56), we obtain the results presented in [23] namely:[5]

$$
\begin{aligned}
a_0 &= 1\\
a_2 &= \tfrac{1}{2}u\\
a_4 &= -\tfrac{1}{8}u^2\\
a_6 &= \tfrac{1}{16}u^3+\tfrac{1}{8}\theta^2(u'^2-2uu^{''})\\
a_8 &= -\tfrac{5}{128}u^4+\tfrac{5}{8}\theta^2\left(u^2u''-\tfrac{1}{2}u'^2u\right)+\tfrac{1}{4}\theta^4\left(u'''u'-uu^{(4)}-\tfrac{1}{2}u^{''2}\right)\\
a_{10} &= \tfrac{7}{256}u^5+\tfrac{35}{64}\theta^2\left(\tfrac{1}{2}u^2u'^2-u^3u''\right)+\tfrac{7}{4}\theta^4\left(\tfrac{3}{4}u^{(4)}u^2+\tfrac{7}{4}u^{''2}u-\tfrac{3}{4}u'^2u''-uu'u'''\right)\\
&+ \tfrac{1}{4}\theta^6\left(u'u^{(5)}+\tfrac{1}{2}u'''^2-uu^6\right)\\
a_{12} &= -\tfrac{21}{1024}u^6+\tfrac{105}{64}\theta^2\left(u^4u''-\tfrac{1}{2}u^3u'^2\right)\\
&+ \tfrac{1}{16}\theta^4\left(147uu''u'^2+\tfrac{189}{2}u^2u'u'''-\tfrac{1029}{4}u^2u^{''2}-63u^3u^{(4)}-\tfrac{105}{8}u'^4\right)\\
&+ \tfrac{1}{4}\theta^6\left(16u^{''3}+9u^2u^{(6)}-27u'u''u'''-\tfrac{45}{2}u'^2u^{(4)}-\tfrac{69}{4}u^{'''2}u+\tfrac{153}{2}uu''u^{(4)}-\tfrac{27}{2}uu'u^{(5)}\right)\\
&+ \tfrac{1}{4}\theta^8\left(u'u^{(7)}+u'''u^{(5)}-u''u^{(6)}-uu^{(8)}-\tfrac{1}{2}{u^{(4)}}^2\right)\\
&\ \ \vdots
\end{aligned}
\tag{57}
$$

with

$$a_{2k+1}=\sum_{s=0}^{2k-1}\theta^s c^s_{2k-1}b^{(s)}_{2k+1-s}=0, \qquad k=0,1,2,3,... \tag{58}$$

Now having derived the explicit expression of $\mathcal{L}^{\frac{1}{2}}$, we are now in position to write the explicit forms of the set of sl_n-Moyal KdV hierarchy. These equations defined as

$$\frac{\partial\mathcal{L}}{\partial t_k}=\{(\mathcal{L}^{\frac{k}{2}})_+,\mathcal{L}\}_\theta, \tag{59}$$

are computed in [23] up to the first three flows t_1,t_3,t_5. We work out these equations by adding

[5]In this work, important explicit computations of the parameters a_{i+1} are presented up to a_{10}. Our calculus (57-58)[20, 21], performed up to a_{12} show some missing terms in the computations of [23] relative to a_{10}

other flows namely t_7 and t_9. We find

$$
\begin{aligned}
u_{t_1} &= u' \\
u_{t_3} &= \tfrac{3}{2}uu' + \theta^2 u''' \\
u_{t_5} &= \tfrac{15}{8}u^2u' + 5\theta^2(u'u'' + \tfrac{1}{2}uu''') + \theta^4 u^{(5)} \\
u_{t_7} &= \tfrac{35}{16}u^3u' + \tfrac{35}{8}\theta^2(4uu'u'' + u'^3 + u^2u''') + \tfrac{7}{2}(uu^{(5)} + 3u'u^{(4)} + 5u''u''')\theta^4 + \theta^6 u^{(7)} \\
u_{t_9} &= 18\theta^6 u'u^{(6)} + \tfrac{651}{8}\theta^4 u'(u'')^2 + \tfrac{315}{128}u^4u' + \tfrac{483}{8}\theta^4 u'^2u''' + \tfrac{315}{16}\theta^2 uu'^3 + \tfrac{189}{4}\ \theta^4 uu^{(4)}u' \\
&+ \tfrac{315}{8}\theta^2u^2u'u'' + \tfrac{315}{4}\theta^4uu'u''' + 63\theta^6u'''u^{(4)} + \tfrac{105}{16}\theta^2u^3u''' + 42\theta^6u^{(5)}u'' \\
&+ \tfrac{63}{8}\theta^4u^2u^{(5)} + \theta^8u^{(9)} + \tfrac{9}{2}\theta^6uu^{(7)}
\end{aligned}
\tag{60}
$$

Some important remarks are in order:

1. The flow parameters t_{2k+1} has the following conformal dimension $[\partial_{t_{2k+1}}] = -[t_{2k+1}] = 2k+1$ for $k = 0, 1, 2, ...,$.

2. A remarkable property of the sl_2-Moyal KdV hierarchy is about the degree of non linearity of the θ-evolution equations (60). We present in the following table the behavior of the higher non-linear terms with respect to the first leading flows $t_1, ..., t_9$ and give the behavior of the general flow parameter t_{2k+1}.

Flows	The higher n.l. terms	Degree of n linearity
t_1	$u^0u' = u'$	0
t_3	$\frac{3}{2}uu'$	1 (quadratic)
t_5	$\frac{15}{2^3}u^2u'$	2 (cubic)
t_7	$\frac{35}{2^4}u^3u'$	3
t_9	$\frac{315}{2^7}u^4u'$	4
...	...	...
t_{2k+1}	$\eta(2k+1)(2k-1)u^ku'$	(k),

(61)

where η is an arbitrary constant.

This result shows among others that the θ-evolution equations (60) exhibit at most a non-linearity of degree (k) associated to a term proportional to $(2k+1)(2k-1)u^k u'$. The particular case $k=0$ corresponds to linear wave equation.

3. The contribution of non-commutativity to the Moyal KdV hierarchy shows a correspondence between the flows t_{2k+1} and the non-commutativity parameters $\theta^{2(k-s)}, 0 \leq s \leq k$. Particularly, the higher term $\theta^{2(k)}$ is coupled to the $k-th$ prime derivative of u_2 namely $u^{(k)}$ while the higher non linear term $\eta(2k+1)(2k-1)u^k u'$ is a θ-independent object as its shown in (60).

4. In analogy with the classical case, once the non linear terms in the θ-evolution equations are ignored, there will be no solitons in the KdV-hierarchy as the latter's are intimately related to non linearity.

4.2 sl_3-Boussinesq Hierarchy

The same analysis used in deriving the sl_2-KdV hierarchy is actually extended to build the sl_3-Boussinesq Moyal hierarchy. The latter is associated to the momentum Lax operator $\mathcal{L}_3 = p^3 + u_2 \star p + u_3$ whose $3-th$ root reads as

$$\begin{aligned} \mathcal{L}^{\frac{1}{3}} &= \Sigma_{i=-1} b_{i+1} \star p^{-i} \\ &= \Sigma_{i=-1} a_{i+1} p^{-i} \end{aligned} \tag{62}$$

in such way that $\mathcal{L}_3 = \mathcal{L}^{\frac{1}{3}} \star \mathcal{L}^{\frac{1}{3}} \star \mathcal{L}^{\frac{1}{3}}$. Explicit computations lead to

$$
\begin{array}{rcl}
b_0 &=& 1 \\
b_1 &=& 0 \\
b_2 &=& \frac{1}{3}u_2 \\
b_3 &=& \frac{1}{3}u_3 - \frac{2}{3}\theta u_2' \\
b_4 &=& -\frac{1}{9}u_2^2 - \frac{2}{3}\theta u_3' + \frac{8}{9}\theta^2 u_2'' \\
b_5 &=& -\frac{2}{9}u_2u_3 + \frac{8}{9}\theta u_2u_2' + \frac{8}{9}\theta^2 u_3'' - \frac{8}{9}\theta^3 u_3''' \\
b_6 &=& \frac{1}{9}\{\frac{5}{9}u_2^3 - u_3^2 + 2\theta(4u_2u_3' + 5u_2'u_3) - 20\theta^2(u_2u_2'' + (u_2')^2) - 8\theta^3 u_3''' + \frac{16}{3}\theta^4 u_2''''\} \\
b_7 &=& \frac{1}{9}\{\frac{5}{3}u_2^2u_3 + 10\theta(u_3u_3' - u_2^2u_2') - \frac{20}{3}\theta^2(5u_2''u_3 + 7u_2'u_3' + u_2u_3''') \\
&& -40\theta^3(3u_2'u_2'' + u_2u_2''') + \frac{16}{3}\theta^4 u_3'''') \\
b_8 &=& \frac{5}{27}(u_2u_3^2 - \frac{2}{9}u_2^4) - \frac{10}{9}\theta(u_2^2u_3' - \frac{7}{3}u_2'u_2u_3) + \frac{20}{81}\theta^2(12u_3^{2\prime} + 31u_2u_2^{2\prime} \\
&& +17u_2^2u_2'' - 15u_3''u_3) + \frac{40}{27}\theta^3(10u_3''u_2' + 13u_2''u_3' + 7u_3u_2''' + 3u_3''u_2) \\
&& +\frac{80}{81}\theta^4(8u_2^4u_2 + 23u_2^{2\prime 2} + 32u_2'u_2''') + \frac{64}{81}\theta^6 u_2^{(6)}
\end{array}
\tag{63}
$$

Similarly, one can easily determine the coefficients a_{i+1} which are also expressed as functions of b_{i+1} and their derivatives. This result is summarized in the expression of $\mathcal{L}^{\frac{1}{3}}$ namely

$$
\begin{array}{rcl}
\mathcal{L}^{\frac{1}{3}} = p &+& \frac{1}{3}u_2p^{-1} \\
&+& \frac{1}{3}\{u_3 - \theta u_2'\}p^{-2} \\
&-& \frac{1}{9}\{u_2^2 + \theta^2 u_2''\}p^{-3} \\
&+& \frac{1}{9}\{-2u_2u_3 + 2\theta u_2'u_2 - \theta^2 u_3'' + \theta^3 u_2'''\}p^{-4} \\
&+& \frac{1}{9}\{\frac{1}{3}\theta^4 u_2^{(4)} + 2\theta u_2'u_3 - u_3^2 + \frac{5}{9}u_2^3\}.p^{-5} \\
&+& \frac{1}{27}\{5u_2^2u_3 - 5\theta u_2^2u_2' + 10\theta^2(u_2'u_3' - u_2''u_3) + \theta^4 u_3^{(4)} - \theta^5 u_2^{(5)}\}.p^{-6} \\
&+& \frac{1}{27}\{\frac{5}{9}u_2(9u_3^2 - 2u_2^3) - 10\theta u_2'u_2u_3 + \frac{5}{3}\theta^2(6u_3'^2 - 6u_3''u_3 + 5u_2^2u_2'' - 2u_2u_2^{2\prime}) \\
&-& 10\theta^3(-u_3''u_2' - u_3u_2''' + 2u_2''u_3') - \frac{10}{3}\theta^4(u_2^{(4)}u_2 + 4u_2'u_2''' - 5u_2''^2) - \frac{1}{3}\theta^6 u_2^{(6)}\}p^{-7} \\
&+& ...
\end{array}
\tag{64}
$$

Furthermore, using the Moyal sl_3-Lax evolution equations

$$\frac{\partial \mathcal{L}}{\partial t_k} = \{(\mathcal{L}^{\frac{k}{3}})_+, \mathcal{L}\}_\theta, \tag{65}$$

that we compute explicitly for $k = 1, 2, 4$ we obtain

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial t_1} &= u_2'p + u_3' - \theta u_2'' \\
\frac{\partial \mathcal{L}}{\partial t_2} &= 2\{u_3' - \theta u_2''\}p - \frac{2}{3}\{u_2u_2' + \theta^2 u_2'''\} \\
\frac{\partial \mathcal{L}}{\partial t_4} &= \frac{4}{3}\{(u_2u_3)' - \theta(u_2''u_2 + u_2'^2) + 2\theta^2 u_3''' - 2\theta^3 u_2^{(4)}\}p \\
&\quad + \frac{4}{3}\{u_3u_3' - \frac{1}{3}u_2^2u_2' - \theta(u_2'u_3' + u_2''u_3) - \theta^2(u_2'u_2'' + u_2u_2''') - \frac{2}{3}\theta^4 u_2^{(5)}\}
\end{aligned} \tag{66}$$

Identifying both sides of the previous equations, one obtain the following first leading evolution equations

$$\begin{aligned}
\frac{\partial}{\partial t_1}u_2 &= u_2' \\
\frac{\partial}{\partial t_1}u_3 &= u_3' \\
\\
\frac{\partial}{\partial t_2}u_2 &= 2u_3' - 2\theta u_2'' \\
\frac{\partial}{\partial t_2}u_3 &= -\frac{2}{3}u_2u_2' - \frac{8}{3}\theta^2 u_2''' + 2\theta u_3'' \\
\\
\frac{\partial}{\partial t_4}u_2 &= \frac{4}{3}\{(u_2u_3)' - \theta(u_2''u_2 + u_2'^2) + 2\theta^2 u_3''' - 2\theta^3 u_2^{(4)}\} \\
\frac{\partial}{\partial t_4}(u_3 - \theta u_2') &= \frac{4}{3}\{u_3u_3' - \frac{1}{3}u_2^2u_2' - \theta(u_2'u_3' + u_2''u_3) - \theta^2(u_2'u_2'' + u_2u_2''') - \frac{2}{3}\theta^4 u_2^{(5)}.
\end{aligned} \tag{67}$$

These equations define what we call the Moyal sl_3 Boussinesq hierarchy. The first two equations are simply linear θ-independent wave equations fixing the dimension of the first flow parameter t_1 to be $[t_1] = -1$.

The non trivial flow of this hierarchy starts really from the second couple of equations associated to t_2. We will discuss in the next section, how its important to deal with the basis of primary conformal fields v_k instead of the old basis u_k. Anticipating this result, one can write the previous couple of equations in term of the spin 3 primary field $v_3 = u_3 - \theta u_2'$ as follows

$$\begin{aligned}
\frac{\partial}{\partial t_2}u_2 &= 2v_3' \\
\frac{\partial}{\partial t_2}v_3 &= -\frac{2}{3}\{u_2u_2' + \theta^2 u_2'''\}
\end{aligned} \tag{68}$$

This couple of equations define the θ-extended Boussinessq equation. Its second-order form is obtained by differentiating the first equation in (68) with respect to t_2 and then using the second

equation. We find

$$\frac{\partial^2}{\partial t_2^2}u_2 = -\frac{4}{3}(u_2u_2' + \theta^2 u_2^{(3)})', \tag{69}$$

Equivalently one may write

$$\begin{pmatrix} u_2 \\ v_3 \end{pmatrix}_{t_2} = -\frac{2}{3}\begin{pmatrix} -3v_2' \\ u_2u_2' + \theta^2 u_2''' \end{pmatrix}. \tag{70}$$

Similarly the third couple of equations (67) can be equivalently written as

$$\begin{aligned} \frac{\partial}{\partial t_4}u_2 &= \frac{4}{3}(u_2v_3 + 2\theta^2 v_3'')' \\ \frac{\partial}{\partial t_4}v_3 &= \frac{4}{3}\{v_3v_3' - \theta^2 u_2u_2''' - \frac{1}{3}(u_2^2u_2' + 2\theta^4 u_2^{(5)})\} \end{aligned} \tag{71}$$

Recall that the classical Boussinesq equation is associated to the sl_3-Lax differential operator

$$\mathcal{L}_3 = \partial^3 + 2u\partial + v_3 \tag{72}$$

with $v_3 = u_3 - \frac{1}{2}u_2'$ defining the spin-3 primary field. This equation which takes the following form

$$u_{tt} = -(auu' + bu^{(3)})', \tag{73}$$

where a, b are arbitrary constants, arises in several physical applications. Initially, it was derived to describe propagation of long waves in shallow water [24]. This equation plays also a central role in $2d$ conformal field theories via its Gelfand-Dickey second Hamiltonian structure associated to the Zamolodchikov w_3 non linear algebra [17, 18].

To close this section note that other flows equations associated to (sl_2)-KdV and (sl_3)-Boussinesq hierarchies can be also derived once some lengthly and hard computations are performed. One can also generalize the obtained results by considering other sl_n integrable hierarchies with $n > 3$.

5 The NC Lax generating technics in the Moyal momentum framework

Using the convention notations and the analysis presented previously and developed in [20, 21] and based also on the results established in [25, 26, 27][6], we present in this section some results related to the Lax representation of noncommutative integrable hierarchies. We perform also consistent algebraic computations, based on the Moyal-momentum analysis, to derive explicit Lax pair operators of some integrable systems in the noncommutative framework.

We underline that the present formulation is based on the (pseudo) momentum operators p^n ad p^{-n} instead of the (pseudo) operators ∂^n and ∂^{-n} used in several works. We note also that the

[6] I am grateful to K. Toda for bringing to my attention ref. 27

obtained results are shown to be compatible with the ones already established in literature [26].

Note also that the notion of integrability of the concerned nonlinear differential equations is defined in the sense that these equations may be linearizable.

To start, let's recall that the sl_n-Moyal KdV hierarchy is defined as

$$\frac{\partial \mathcal{L}}{\partial t_k} = \{(\mathcal{L}^{\frac{k}{2}})_+, \mathcal{L}\}_\theta. \tag{74}$$

Explicit computations related to these hierarchies are presented in [20, 21]. Working these hierarchies, we was able to derive, among others, for the sl_2 case up to the flow t_9, the following KdV-hierarchy equations

$$\begin{aligned} u_{t_1} &= u', \\ u_{t_3} &= \tfrac{3}{2}uu' + \theta^2 u''', \\ u_{t_5} &= \tfrac{15}{8}u^2u' + 5\theta^2(u'u'' + \tfrac{1}{2}uu''') + \theta^4 u^{(5)}, \\ u_{t_7} &= \tfrac{35}{16}u^3u' + \tfrac{35}{8}\theta^2(4uu'u'' + u'^3 + u^2u''') + \tfrac{7}{2}(uu^{(5)} + 3u'u^{(4)} + 5u''u''')\theta^4 + \theta^6 u^{(7)}, \\ &\dots \end{aligned} \tag{75}$$

Actually this construction which works well for the sl_2-KdV hierarchy is generalizable to higher order KdV hierarchies, namely the sl_n-KdV hierarchies.

The basic idea of the Lax formulation consists first in considering a noncommutative integrable system which possesses the Lax representation such that the following noncommutative Moyal bracket

$$\{L, T + \partial_t\}_\theta = 0, \tag{76}$$

is equivalent to the noncommutative differential equation that we consider from the beginning and that is nonlinear in general with $\partial_t \equiv \frac{\partial}{\partial t}$.
Equation (76) and the associated pair of operators (L, T) are called the Lax differential equation and the Lax pair, respectively. The differential operator L defines the integrable system which we should fix from the beginning.

Note that the way with which ones to writes the Lax equation as in (76) is equivalent to that in (74) namely

$$\{L, T + \partial_t\}_\theta \equiv \{L, (\mathcal{L}^{\frac{k}{2}})_+ + \partial_{t_k}\}_\theta = 0, \tag{77}$$

where the operator T is the analogue of $(\mathcal{L}^{\frac{k}{2}})_+$ describing then an operator of conformal spin k.

This equation, written in terms of the function $u(x,t)$, is in general a non linear differential

equation belonging to the ring $\widehat{\Sigma}^{(0,0)}_{k+2}$. In the present case of sl_2-KdV systems we have $k=3$.

As it's shown in [26], the meaning of Lax representations in noncommutative spaces would be vague. However, they actually have close connections with the bi-complex method [28] leading to infinite number of conserved quantities, and the (anti)-self-dual Yang-Mills equation which is integrable in the context of twistor descriptions and ADHM constructions [29, 30].

Now, let us apply the noncommutative Lax-pair generating technique. Usually, it's a method to find a corresponding T-operator for a given L-operator. Finding the operator T satisfying (76) is not an easy job in the general case. For this reason, one have to make some constraints on the operator T namely:

Ansatz for the operator T:

$$T = p^n \star L^m + T', \tag{78}$$

where p^n are momentum operators acting on arbitrary function $f(x,p)$ as shown in *section 2*. Note by the way that the notation T' have nothing to do with the prime derivative. With the previous ansatz, the problem reduces to that for the T'-operator which is determined by hand so that the Lax equation should be a differential equation belongings to the ring $\widehat{\Sigma}^{(0,0)}$.
The best way to understand what happens for the general case, is to focus on the following examples:
Example 1: The sl_2-noncommutative KdV system.

The L-operator for the noncommutative KdV equation is given, in the momentum space configuration, by

$$L = p^2 + u(x,t), \tag{79}$$

with

$$L \in \widehat{\Sigma}^{(0,2)}_2 / \widehat{\Sigma}^{(1,1)}_2, \tag{80}$$

where $\widehat{\Sigma}^{(1,1)}_2$ is the one dimensional subspace generated by objects of type $\xi_1(x,t)\star p$ and $\xi_1(x,t)$ is an arbitrary function of conformal spin 1.

Reduced to $n = 1 = m$, for the noncommutative sl_2 KdV system, the ansatz (25) can be written as follows[7]

$$T = p \star L + T'. \tag{81}$$

The operator T in this case ($k=3$), is shown to behaves as $(\mathcal{L}^{\frac{3}{2}})_+$ with $\partial_{t_3} \equiv \frac{\partial}{\partial t_3}$.

[7] One can also introduce the following definition: $T \equiv (p\star L)_s + T'$, with the convention $(p\star L)_s \equiv \frac{(p\star L + L\star p)}{2}$ describing the symmetrized part of the operator $p\star L$

Simply algebraic computations give

$$\{L, T'\}_\theta = u'\mathbf{p}^2 - 2\theta u''\mathbf{p} + \frac{\dot{u}}{2\theta} + (uu' + \theta^2 u'''), \tag{82}$$

Next, our goal is to be able to extract the Lax differential equation, namely, the noncommutative KdV equation. Before that, we have to make a projection of the operator $\{L, T'\}_\theta$ on the ring $\widehat{\Sigma}_{3+2}^{(0,0)}$. This projection is equivalent to cancel the effect of the terms of momentum in (82), namely the term $u'\mathbf{p}^2$ and $2\theta u''\mathbf{p}$. To do that, we have to consider the following property:

Ansatz for T':

$$T' = X \star p + Y, \tag{83}$$

where X and Y are arbitrary functions on u and its derivatives.

Next, performing straightforward computations, with $T' = Xp - \theta X' + Y$ lead to

$$\{L, T'\}_\theta = 2X'\mathbf{p^2} + (\{u, X\}_\theta - 2\theta X'' + 2Y')\mathbf{p} + (-Xu' - \theta\{u, X'\} + \{u, Y\}) \tag{84}$$

Identifying (82) and (84), leads to the following constrains equations

$$X = \frac{1}{2}u_2 + a, \tag{85}$$

$$Y = -\frac{1}{2}\theta u_2' + b, \tag{86}$$

with the following nonlinear differential equation

$$-\frac{\dot{u}}{2\theta} = \frac{3}{2}uu' + \theta^2 u'''. \tag{87}$$

where the constants a and b are to be omitted for a matter of simplicity. The last equation is noting but the sl_2 KdV equation. This noncommutative equation contains also a non linear term $\frac{3}{2}uu'$.

We have to underline that the sl_2 noncommutative KdV equation obtained through this Lax method belongs to the same class of the KdV equation derived in [20, 21] namely

$$\dot{u} = \frac{3}{2}uu' + \theta^2 u'''. \tag{88}$$

In fact, performing the following scaling transformation $\partial_{t_3} \to -2\theta\partial_{t_3}$ we recover exactly (88). The term $\frac{1}{2\theta}$ appearing in (87) as been the coefficient of the evolution part $\dot{u}_2$ of the NC KdV equation can be simply shifted to one due to consistency with respect to the classical limit $\theta_l \sim \frac{1}{2}$.

To summarize, the momentum Lax pair operators, associated to the noncommutative sl_2-KdV system, are explicitly given by

$$L_{KdV} = p^2 + u_2(x, t), \tag{89}$$

and

$$T_{KdV} = \mathbf{p}^3 + \frac{3}{2}\mathbf{p} \star u_2(x,t) - \frac{3}{2}\theta u_2'(x,t); \tag{90}$$

with $T' = \frac{1}{2}\mathbf{p} \star u_2(x,t) - \frac{3}{2}\theta u_2'(x,t)$

Note that, the same results can obtained by using the Gelfand-Dickey (GD) formulation based on formal (pseud) differential operators $\partial^{\pm n}$ instead of the Moyal momentum ones.

This first example shows, among others, the consistency of the Moyal momentum formulation in describing integrable systems and the associated Lax pair generating technics in the same way as the successful GD formulation [16].

Example 2: The Noncommutative Burgers Equation *à* la Moyal

Let us apply the same noncommutative Lax technics, presented previously, to derive the non-commutative version of the Burgers equation. Actually, our interest in this equation comes from the several important properties that are exhibited in the commutative case. Before going into applying the noncommutative Lax technics, let's first recall some few known properties of the standard Burgers equation.

P1: The Burgers equation is defined on the $(1+1)$- dimensional space time. In the standard pseudo-differential operator formalism, this equation is associated to the following L-operator

$$L_{Burgers} = \partial_x + u_1(x,t) \tag{91}$$

where the function u_1 is of conformal spin one. Using our convention notations, we can set $L \in \widehat{\Sigma}_1^{(0,1)}$.

P2: With respect to the previous L-operator, the non linear differential equation of the Burgers equation is given by

$$\dot{u}_1 + \alpha u_1 u_1' + \beta u_1'' = 0, \tag{92}$$

where $\dot{u} = \frac{\partial u}{\partial t}$ and $u' = \frac{\partial u}{\partial x}$. The dimensions of the underlying objects are given by $[t] = -2 = -[\partial_t]$, $[x] = -1$ and $[u] = 1$.

P3: On the commutative space-time, the Burgers equation can be derived from the Navier-Stokes equation and describes real phenomena, such as the turbulence and shock waves. In this sense, the Burgers equation draws much attention amongst many integrable equations.

P4: It can be linearized by the Cole-Hopf transformation [31]. The linearized equation is the diffusion equation and can be solved by Fourier transformation for given boundary conditions.

P5: The Burgers equation is completely integrable [32].

Now, we are ready to look for the noncommutative version of the Burgers equation. For that, we consider the L-operator of this equation in the Moyal momentum language, namely

$$L = p + u_1(x,t). \tag{93}$$

dealing, as noticed before, to the space $\widehat{\Sigma}_1^{(0,1)}$. This is a local differential operator of the generalized n-KdV hierarchy's family ($n = 1$), obtained by a truncation of a noncommutative pseudo momentum operator of KP hierarchy type

$$L = p + u_1(x,t) + u_2(x,t) \star p^{-1} + u_3(x,t) \star p^{-2} + ..., \tag{94}$$

of the space $\widehat{\Sigma}_1^{(-\infty,1)}$. The local truncation is simply given by

$$\widehat{\Sigma}_1^{(-\infty,1)} \rightarrow \widehat{\Sigma}_1^{(0,1)} \equiv [\widehat{\Sigma}_1^{(-\infty,1)}]_+ \equiv \widehat{\Sigma}_1^{(-\infty,1)} / \widehat{\Sigma}_1^{(-\infty,-1)}, \tag{95}$$

or equivalently

$$L_1(u_i) = p + \Sigma_{i=0}^{\infty} u_i \star p^{1-i} \rightarrow p + u_1 \equiv [L_1(u_i)]_+, \tag{96}$$

where the symbol $(X)_+$ defines the local part (only positive powers of p) of a given pseudo operator X.

The noncommutative Burgers equation is said to have the Lax representation if there exists a suitable pair of operators (L, T) so that the Lax equation

$$\{p + u_1, T + \partial_t\}_\theta = 0, \tag{97}$$

reproduces the noncommutative version of the Burgers non linear differential equation. Following the same steps developed previously for the sl_2 noncommutative KdV systems, we consider the following ansatz for the operator T:

$$T = p \star L + T', \tag{98}$$

or

$$T = p^2 + u_1 p + \theta u_1' + T'. \tag{99}$$

Then, performing straightforward computations, the noncommutative Burgers Lax equation (97) reduces to

$$\{p + u, T'\}_\theta = u'\mathbf{p} + (uu' - \theta u'' + \frac{\dot{u}}{2\theta}) \tag{100}$$

Next, one have also the go through a constraint equation for the operator T', namely the

Ansatz for T':

$$T' = A \star p + B, \tag{101}$$

where A and B are arbitrary functions for the moment.With this new ansatz for T', we have

$$\{p+u,T'\}_\theta = (A' + \{u,A\}_\theta)\mathbf{p} + (-Au' - \theta A'' - \theta\{u,A'\}_\theta + B' + \{u,B\}_\theta) \tag{102}$$

Identifying (100) and (102) leads to the following constraints equations

$$u' = A' + \{u,A\}_\theta) \tag{103}$$

and

$$(u+A)u' + \frac{\dot{u}}{2\theta} = B' + \{u,B\}_\theta + \theta\{A',u\} + \theta(u'' - A'') \tag{104}$$

A natural solution of the first constraint equation (103) is $A = u$. This leads to a reduction of (104) to

$$2uu' + \frac{\dot{u}}{2\theta} = B' + \{u,B\} \tag{105}$$

Actually this is the noncommutative Burgers equation, which is also the projection of the Lax equation (97) to the ring of vanishing degrees in momenta namely the space $\widehat{\Sigma}_1^{(0,0)}$.

Since $\{u,\partial_t\}_\theta = -\frac{\dot{u}}{2\theta}$, a non trivial solution of the parameter B in equation (105) consists in setting $B \equiv u^2 - \frac{\partial}{\partial t}$. But, since this nontrivial solution of B masks the noncommutative Burgers equation, it's a non desirable thing.

Remarking also that $[B] = 2$, we use this dimensional arguments and set

$$B = \xi u' + \eta u^2 \tag{106}$$

with ξ and η are arbitrary coefficient numbers. Putting this expression into (105) gives the final expression of the noncommutative Burgers equation namely

$$\frac{\dot{u}}{2\theta} + 2(1-\eta)uu' - \xi u'' = 0 \tag{107}$$

whose Lax pair in the noncommutative Moyal momentum formalism are explicitly given by

$$L_{Burgers} = p + u_1(x,t) \tag{108}$$

and

$$T_{Burgers} = p^2 + 2u_1(x,t)p + \eta u_1^2(x,t) + \xi u_1'(x,t) \tag{109}$$

6 Noncommutative Burgers-KdV mapping

This section will be devoted to another significant aspect of the noncommutative integrable models. The principal focus, for the moment, is on the models discussed previously namely the noncommutative KdV and Burgers systems. In *section 5* we discussed the integrability of these

two nonlinear systems and we noted that they are indeed integrable and this property is due to the existence of definite Lax pair operators (L, T) for each of the two models. Such existence implies the linearization of the models automatically.

A crucial question which arises now is to know if it there is a possibility to establish a mapping between the two Systems. The idea to connect the two models is originated from the fact that integrability for the KdV system both in commutative and noncommutative spaces is something natural due to the possibility to connect with $2d$ conformal symmetry. We think that the strong backgrounds of conformal symmetry can help to shed more lights about integrability of the noncommutative Burgers systems if one know how to establish such a connection.

On the other hand, it is clear that these models are different due to the fact that for the noncommutative KdV system the Lax operator as well as the associated field $u_2(x,t)$ are of conformal weights 2, whereas for the Burgers system, L and u_1 are of weight 1.

Our goal is to study the possibility of transition between the two spaces $\widehat{\Sigma}_2^{(0,2)}/\widehat{\Sigma}_2^{(1,1)}$ and $\widehat{\Sigma}_1^{(0,1)}$ corresponding respectively to the two models. This transition, once it exists, should leads to extract more informations on these noncommutative models and also on their integrability.

To start, let's consider the following property

Proposition 1:
Lets consider the Burgers momentum operator $L_{Burgers}(u_1) = p + u_1 \in \widehat{\Sigma}_1^{(0,1)}$. For any given sl_2 noncommutative KdV operator $L_{KdV}(u_2) = p^2 + u_2(x,t)$ belongings to the space $\widehat{\Sigma}_2^{(0,2)}/\widehat{\Sigma}_2^{(1,1)}$, one can define the following mapping

$$\widehat{\Sigma}_1^{(0,1)} \hookrightarrow \widehat{\Sigma}_2^{(0,2)}/\widehat{\Sigma}_2^{(1,1)}, \tag{110}$$

in such away that

$$L_{Burgers}(u_1) \rightarrow L_{KdV}(u_2) \equiv L_{Burgers}(u_1) \bigotimes L_{Burgers}(-u_1). \tag{111}$$

We know that the space $\widehat{\Sigma}_2^{(0,2)}$ of momentum operators of conformal spin $s = 2$ is different from the one of momentum operators of conformal spin $s = 1$ namely $\widehat{\Sigma}_1^{(0,1)}$. What we are assuming in this proposition is a strong constraint leading to connect the two spaces. This constraint is also equivalent to set

$$\widehat{\Sigma}_2^{(0,2)}/\widehat{\Sigma}_2^{(1,1)} \equiv \widehat{\Sigma}_1^{(0,1)} \bigotimes \widehat{\Sigma}_1^{(0,1)} \tag{112}$$

Next we are interested in discovering the crucial key behind the previous proposition. For this reason, we underline that this mapping is easy to highlight through the noncommutative

analogue of the well known Miura transformation

$$L_{KdV} = p^2 + u_2 = (p^1 + u_1) \star (p^1 - u_1) \tag{113}$$

giving rise to

$$u_2 = -u_1^2 - 2\theta u_1^{'}. \tag{114}$$

This is an important property since one have the possibility to realize the KdV sl_2 noncommutative field u_2 in term of the Burgers field u_1, its derivative u_1' and of the θ-parameter. This realizations shows among other an underlying nonlinear behavior in the KdV noncommutative field u_2 given by the quadratic term u_1^2.

However, the *proposition 1* can have a complete and consistent significance only if one manages to establish a connection between the noncommutative differential equations associated to the two systems. Arriving at this stage, note that besides the principal difference due to conformal spin, we stress that the two nonlinear evolutions equations of KdV

$$-\frac{1}{2\theta}\frac{\partial u_2}{\partial t_3} = \frac{3}{2}uu' + \theta^2 u'''. \tag{115}$$

and of Burgers

$$\frac{1}{2\theta}\frac{\partial u_1}{\partial t_2} + 2(1-\eta)uu' - \xi u'' = 0 \tag{116}$$

are distinct by a remarkable fact that is the KdV flow $t_{KdV} \equiv t_3$ and the Burgers one $t_{Burgers} \equiv t_2$ have different conformal weights: $[t_{KdV}] = -3$ whereas $[t_{Burgers}] = -2$.

In order to be consistent with the objective of the *proposition 1*, based on the idea of the possible link between the two noncommutative integrable systems, presently we are constrained to circumvent the effect of proper aspects specific to both the equations and consider the following second property:

Proposition 2:

By virtue of the Burgers-KdV mapping and dimensional arguments, the associated flow are related through the following ansatz

$$\partial_{t_2} \hookrightarrow \partial_{t_3} \equiv \partial_{t_2}.\partial_x + \alpha\partial_x^3 \tag{117}$$

for an arbitrary parameter α.

With respect to the assumption (117), relating the two evolution derivatives ∂_{t_2} and ∂_{t_3} belongings to Burgers and KdV hierarchies respectively, one should expect some strong constraint on the Burgers noncommutative differential equation (116). Such constraint is important since

one needs to fix the arbitrary coefficients ξ and η which are still arbitrary.
We have to identify the following three differential equations

$$\begin{aligned}\partial_{t_3}u_2 &= \tfrac{3}{2}u_2u_2' + \theta^2 u_3''',\\ &= -2u_1\partial_{t_3}u_1 - 2\theta\partial_{t_3}u_1',\\ &= \partial_{t_2}u_2' + \alpha u_2'''.\end{aligned} \tag{118}$$

Some explicit results:

Setting for a matter of simplicity the Burgers equation as $\partial_{t_2}u_1 = Au_1u_1' + Bu_1''$ with $A = 4\theta(\eta - 1)$ and $B = 2\theta\xi$, and performing explicit computation, rising from the identification of the previous system of equations (118), we find the following results:

$$\begin{aligned}\partial_{t_3}u_2 &= 3u_1^3u_1' + 6\theta u_1'^2u_1 + 3\theta u_1''u_1^2 - 2\theta^2u_1'''u_1 - 2\theta^3u_1''''\\ &= -2Au_1'^2u_1 - 6\theta Au_1''u_1' - 2Au_1^2u_1'' - 2\theta u_1''''(B+\alpha) - 2u_1u_1'''(\alpha + A\theta + B)\\ &= -4Au_1'^2u_1 - 2u_1''u_1'(B + 3\alpha + 3A\theta) - 2u_1'''u_1(B + \alpha + A\theta) - 2Au_1''u_1^2 - 2\theta u_1''''(B+\alpha)\end{aligned} \tag{119}$$

These expressions, once are simplified, lead to the following constraint equation

$$Au_1u_1' + (B + 3\alpha)u_1'' = 0. \tag{120}$$

Putting this constraint equation into the noncommutative Burgers equation (116) give the following equation

$$\partial_{t_2}u_1 = -3\alpha u_1'', \tag{121}$$

which is a linear differential equation. This is an impressing result deserving special interest since one have the possibility to convert a nonlinear differential equation to a linear one.

We will try now to explain the introduced Burgers-KdV mapping and the induced linearizability property in connection with our guess of a hidden $2d$-conformal symmetry. First of all note that the conformal symmetry in the framework of noncommutative KdV hierarchy is related in general to the sl_n-symmetry as it's explicitly shown in *section 3*, see also [20] for more details. Referring to these algebraic backgrounds, we can make contact with Burgers-KdV mapping. In fact, we have to remark that the Burgers u_1-current issued from the Miura like equation (113) and satisfying (114) can be identified, by virtue of (45-46), with the Liouville Lorentz scalar field ϕ as follows

$$u_1(x,t) \equiv \phi' \tag{122}$$

with $\phi' \equiv \partial\phi \equiv \frac{\partial}{\partial x}\phi(x,t)$ while the noncommutative KdV potential $u_2(x,t)$ satisfying (114) can be then identified with the conformal current T given by (47). Using all these equations one can

actually interpret the constraint equation (120), induced from the Burgers-KdV mapping, as been the equation of conservation of the noncommutative KdV potential u_2. In fact, making an analogy with $2d$ conformal field theory construction and using (114) the noncommutative KdV current read in terms of the u_1-Burgers current as $T \equiv u_2(x,t) = -u_1^2 - 2\theta u_1^{'}$ which coincides also with (47) once we introduce the θ-Liouville field ϕ. Requiring the conservation property for this current, namely

$$u_1 u_1' + \theta u_1^{''} = 0 \tag{123}$$

reproduce exactly the constraint equation (120) induced from the Burgers-KdV mapping with the following fixation of the parameters ξ and η

$$\begin{aligned} \xi &= \tfrac{1}{4\theta} + 1 \\ \eta &= \tfrac{\theta - 3\alpha}{2\theta} \end{aligned} \tag{124}$$

On the other hand, the constraint equation (120), which we consider now as been a conservation law, is also equivalent to set

$$\phi = -\theta log \phi'', \tag{125}$$

giving rise then to the following θ-Liouville like differential equation

$$\phi'' = Kexp(-\frac{1}{\theta}\phi) \tag{126}$$

Based on the analogy with (48) the constant can be simply chosen equal to $K = \frac{2}{\theta}$

7 Concluding Remarks

1. The results obtained for the Moyal Momentum algebra[8] are applied to study some properties of sl_2-KdV and sl_3-Boussinesq integrable hierarchies. Our contributions to this study consist in extending the results found in literature by increasing the order of computations a fact which leads us to discover more important properties as its explicitly shown in [20, 21].

2. We have presented a systematic study of the Moyal momentum algebra that we denote in our convention notation as $\widehat{\Sigma}_\theta$. This is the huge space of momentum Lax operators of arbitrary conformal spin m,$m \in Z$ and arbitrary higher and lowest degrees (r, s) reading as

$$\tilde{\mathcal{L}}_m^{(r,s)}(u) = \sum_{i=r}^{s} p^i \star u_{m-i} \tag{127}$$

3. We studied the algebraic properties of $\widehat{\Sigma}_\theta$ and its underlying sub-algebras $\widehat{\Sigma}_m^{(r,s)}$ and show that among all these spaces only the subspace $\widehat{\Sigma}_0^{(-\infty,1)}$ which defines a Lie algebra structure with respect to the Moyal bracket.

4. The particular sub-algebra $sl_n - \widehat{\Sigma}_n^{(0,n)}$ built out of the sl_n momentum Lax operators $\tilde{\mathcal{L}}_n^{(0,n)}(u) = \sum_{i=0}^{n} p^i \star u_{n-i}$, with $u_0 = 1$ and $u_1 = 0$, is applied to field theory building. Indeed, using the properties of this sub-algebra we were able to construct the θ-Liouville conformal model

$$\partial\bar{\partial}\phi = \frac{2}{\theta} e^{-\frac{1}{\theta}\phi} \tag{128}$$

and its sl_3-Toda extension.

$$\begin{aligned} \partial\bar{\partial}\phi_1 &= A e^{-\frac{1}{2\theta}(\phi_1 + \frac{1}{2}\phi_2)} \\ \partial\bar{\partial}\phi_2 &= B e^{-\frac{1}{2\theta}(\phi_1 + 2\phi_2)} \end{aligned} \tag{129}$$

5. We show also that the central charge, a la Feigin-Fuchs, associated to the spin-2 conformal current of the θ-Liouville model is given by

$$c_\theta = (1 + 24\theta^2) \tag{130}$$

6. We derived also the noncommutative sl_2-KdV and sl_3-Boussinesq hierarchies and write their associated θ-flows. The NC KdV equation is given by

$$\dot{u} = \frac{3}{2} u u' + \theta^2 u''', \tag{131}$$

while the NC Boussinesq equation given by the couple of equations

$$\begin{pmatrix} u_2 \\ v_3 \end{pmatrix}_{t_2} = -\frac{2}{3} \begin{pmatrix} -3v_2' \\ u_2 u_2' + \theta^2 u_2''' \end{pmatrix} \tag{132}$$

[8] The appellation of Moyal momentum algebra introduced for the first time by Das and Popowicz, see [33, 34]

7. Besides the above established results in the Moyal momentum framework, we tried also to understand much more the meaning of integrability of noncommutative nonlinear systems. The principal focus was on the NC KdV and NC Burgers systems. For this reason, a first contribution was to derive these two equations using the above systematic algebraic formulation in the context of noncommutative Lax pair building (*Section 5*).

8. Concerning the noncommutative derived KdV system, this is an integrable model due to the existence of a noncommutative Lax pair operators (L, T). This existence is an important indication of integrability, but we guess that the realistic source of integrability of this model is the underlying conformal symmetry, shown to play a similar role as in the commutative case. The derived NC KdV equation $-\frac{1}{2\theta}\partial_{t_3}u_2 = \frac{3}{2}uu' + \theta^2 u'''$, through the NC Lax pair building, is equivalent to the one derived in *Section 4* [20, 21] $\partial_{t_3}u_2 = \frac{3}{2}uu' + \theta^2 u'''$ once the scaling transformation $\partial_{t_3} \rightarrow -2\theta\partial_{t_3}$ is considered.

9. Concerning the noncommutative Burgers system (107) that we consider in the second example, it's also an integrable equation whose Lax pair operators are explicitly derived (108-109). Note for instance that the Burgers Lax operator $L_{Burgers} = p + u_1(x,t)$ is a momentum operator belongings to the space $\widehat{\Sigma}_1^{(0,1)}$.

10. As a first checking of integrability for the noncommutative Burgers system, we proceeded to an explicit derivation of the Lax pair operators (L, T) giving the following differential equation $2uu' + \frac{\dot{u}}{2\theta} = B' + [u, B]$. The idea is to solve this equation in terms of the coefficient parameter B such that it can reduces to the non linear Burgers equation belonging to the space $\widehat{\Sigma}_3^{(0,0)}$. Solving this equation give explicitly the requested Lax operator.

11. We should also underline that the importance of this study comes also from the fact that the results obtained in the framework of Moyal momentum are similar to those coming by using the Gelfand-Dickey pseudo operators approach [26, 27].

12. Concerning the possibility to establish a correspondence between the NC KdV and NC Burgers systems. Actually, we succeeded to build a mapping leading to transit from the NC Burgers system to the NC KdV system. The main line of this mapping deals with the following ansatz $\partial_{t_2} \hookrightarrow \partial_{t_3} \equiv \partial_{t_2}\partial_x + \alpha\partial_x^3$ for an arbitrary parameter α. This ansatz is important for several reasons, we give here bellow some of them:

a.) It implies the linearization of the Burgers equation.

b.) It helps to fix the arbitrary NC Burgers coefficients ξ and η through a strong linearizability constraint $Au_1u_1' + (B + 3\alpha)u_1'' = 0$.

c.) From the conformal field theory point of view, this constraint equation is nothing but the analogue of the conservation law of the conformal current.

d.) The idea behind this mapping, as it's shown in *Section 6*, is that for the noncommutative KdV model, the problem of integrability does not arise in the same way as it's for the noncommutative Burgers equation. The first one is mapped to conformal field theory through the Liouville model. This is in fact a strong indication of integrability in favor of noncommutative KdV equation which could help more to understand the noncommutative Burgers system. We believe that the considered mapping might help to bring new insights towards understanding the integrability of noncommutative $2d$-systems.

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